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Equations Governing Surface Streamlines

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Introduction

THE idealized two-dimensional flow rarely occurs in practice. Real situations always have regions of three-dimensional flow. The study of separation and attachment in three-dimensional flows focuses upon the pattern of streamlines and vortex lines on the surface. Previous discussions of the possible configurations of streamlines at separation have used a geometric basis. That is, they essentially discuss trajectories in a plane that might satisfy an arbitrary equation. This Note will present physical equations which surface streamlines must obey. These equations reveal in a general way how properties of the flowfield influence the surface streamline patterns.

We consider the incompressible viscous flow over a solid body. Streamlines on the surface are well-defined, although, since the velocity on the wall is zero, there is some reluctance to call them streamlines (see Fig. 1). They have a definite direction which may be found as the direction of the velocity vector in the limit as the wall is approached. This is also the direction of the shear stress for a Newtonian fluid, and the wall streamlines are sometimes called shear stress lines. The coincidence of the direction of the wall shear stress vector $n \cdot \tau$ (n is the unit normal perpendicular to the wall and τ is the viscous stress tensor), and the streamline direction $\tan\theta = \lim_{x_2 \to 0} (v_3/v_1)$, is proved by expanding the velocity in a Taylor series about the wall:

$$v_1 = \frac{\partial v_1}{\partial x_2} x_2 + O[x_2^2], \quad v_2 = \frac{\partial^2 v_2}{\partial x_2^2} x_2^2 + O[x_2^3],$$

$$v_3 = \frac{\partial v_3}{\partial x_3} x_2 + O[x_2^2]$$

The normal component v_2 is of order x_2^2 because the continuity equation requires $\partial v_2/\partial x_2 = 0$ at the wall. Direct substitution shows the viscous stress and streamline to have the same direction.

The wall also contains a set of vortex lines. They lie on the surface, i.e., $\omega_2 = 0$, as a result of the no-slip condition. By direct computation one may prove that the following relation is true at the wall:

$$\mu\omega \times n = n \cdot \tau$$

Thus, a fluid which obeys a Newtonian viscosity law will produce wall vortex lines that are perpendicular to the wall streamlines. The magnitude of the vorticity is also proportional to the wall shear stress. Lighthill's[1] discussion of wall streamlines and vortex lines emphasizes that separation in three-dimensional flows has only isolated points where the vorticity vanishes and the wall streamlines branch. The idea of a line of separation where $\omega = 0$ is an idealization for a strictly two-dimensional flow.

A differential equation which governs surface streamlines will be derived in this Note. The equation is a Poisson type with pressure and vorticity source terms. The equation should be useful in the study of three-dimensional viscous flows, three-dimensional boundary layers, and separation phenomena. Two-dimensional flows satisfy the equation in a trivial manner. The form appropriate for three-dimensional boundary layer will be presented first. Then the derivation will be generalized to arbitrary viscous flows.

Derivation for a Boundary Layer

Consider the body shown in Fig. 2 and assume the surface is regular with a continuously turning normal vector. A coordinate system is chosen so that the wall is the surface $x_2 = 0$. The flowfield vorticity is ω, and the wall vorticity is given the special symbol $\Omega(x_1, x_3)$,

$$\Omega(x_1,x_3) \equiv [\omega_1(x_1,0,x_3), 0, \omega_3(x_1,0,x_3)] = \omega(x_1,0,x_3) \quad (1)$$

The wall vorticity has only two components and will be viewed as a two-dimensional vector.

The momentum equation for incompressible flow may be written as

$$\frac{\partial v}{\partial t} + \nabla\left(\frac{1}{2}v^2 + \frac{p}{\rho}\right) = -v \times \omega - \nu \nabla \times \omega \quad (2)$$

On the wall this becomes

$$\nabla p = -\mu \nabla \times \omega \quad (3)$$

We are only interested in the normal component of this equation

$$n \cdot \nabla p = \mu(n \cdot \nabla \times \omega)$$

$$\frac{1}{h_2}\frac{\partial p}{\partial x_2} = -\frac{\mu}{h_1 h_3}\left[\frac{\partial}{\partial x_1}(h_3\omega_3) - \frac{\partial}{\partial x_3}(h_1\omega_1)\right]$$

or

$$n \cdot \nabla p = -\mu(n \cdot \hat{\nabla} \times \Omega) \quad (4)$$

where $\hat{\nabla}$ is a surface operator.

At this stage, the boundary-layer approximation $\partial p/\partial x_2 = 0$ (this is not a good assumption near separation) is used to conclude that Ω is itself an irrotational vector:

$$0 = \hat{\nabla} \times \Omega \quad (5)$$

Since Ω is irrotational, and the Stokes theorem applies in the surface (Ref. 2, p. 95), there exists a vorticity potential $\xi(x_1, x_3)$ such that

$$\Omega = \hat{\nabla}\xi \quad (6)$$

Received July 26, 1977; revision received Nov. 18, 1977.
Index categories: Hydrodynamics; Viscous Nonboundary-Layer Flows.
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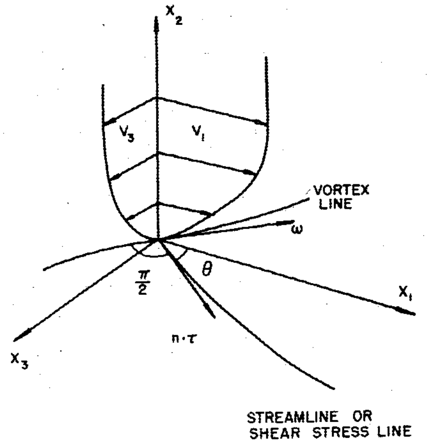

Fig. 1 Surface streamlines are perpendicular to wall vortex lines.

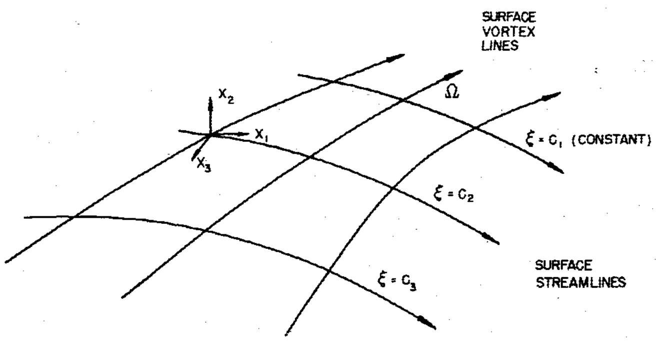

Fig. 2 Streamline and vortex lines on a solid wall.

Lines of $\hat{\xi}$ = constant are perpendicular to $\hat{\nabla}\hat{\xi}$ and hence will be perpendicular to $\boldsymbol{\Omega}$. Since streamlines and vortex lines are also perpendicular, the ξ lines coincide with the wall streamlines. In this sense, we may regard $\hat{\xi}$ as a wall "streamfunction." Numerical values of $\hat{\xi}$ are related to the vorticity (and, hence, the shear stress) by the preceding equation.

The differential equation which governs $\hat{\xi}$ is found simply by noting that ω must be solenodial.

$$\nabla \cdot \omega = 0 \tag{7}$$

Breaking this into its components and evaluating at the wall, we get

$$\frac{1}{h_1 h_3}\frac{\partial}{\partial x_1}(h_3\Omega_1) + \frac{1}{h_1 h_3}\frac{\partial}{\partial x_3}(h_1\Omega_3) = \frac{1}{h_2}\frac{\partial \omega_2}{\partial x_2}$$

$$+ \frac{\omega_1}{h_1 h_2}\frac{\partial h_2}{\partial x_1} + \frac{\omega_3}{h_2 h_3}\frac{\partial h_3}{\partial x_3}$$

$$\hat{\nabla} \cdot \boldsymbol{\Omega} = -\boldsymbol{n} \cdot (\mathrm{d}\omega/\mathrm{d}n)_{x_2=0} \tag{8}$$

This last form is discussed in Ref. 2, p. 209. Substituting from Eq. (6) gives the final form:

$$\hat{\nabla}^2 \hat{\xi} = -\boldsymbol{n} \cdot (\mathrm{d}\omega/\mathrm{d}n)_{x_2=0} \tag{9}$$

Our wall vorticity potential satisfies a Poisson equation where the source term comes from the boundary layer. The component ω_2 is the vorticity in the direction normal to the wall and is not negligible in three-dimensional boundary layers. Although $\omega_2 = 0$ at the wall, the flux of ω_2 across the wall, $\partial\omega_2/\partial x_2$, acts as a source term in preceding equation.

Three-dimensional boundary-layer calculations that employ boundary conditions on the velocity components do not necessarily produce surface streamlines obeying Eq. (9). Note also that the equation is elliptic and that changing $\partial\omega_2/\partial x_2$ at one location influences the surface streamlines all over the wall. This characteristic contrasts with the parabolic boundary-layer equations where only downstream influence occurs.

General Three-Dimensional Flows

This case is different in that the normal pressure gradient is no longer assumed to be zero. This stops us from using the potential in Eq. (6) and a new approach must be taken. Any two-dimensional vector such as $\boldsymbol{\Omega}$ (a function of two independent variables) is a "complex lamellar" vector, because it is perpendicular to its own curl (Ref. 3, p. 64),

$$\boldsymbol{\Omega} \cdot \hat{\nabla} \times \boldsymbol{\Omega} = 0 \tag{10}$$

When this condition is satisfied, there exists an integrating factor $\eta(x_1, x_3)$ so that $\boldsymbol{\Omega}/\eta$ has a potential. We use ξ as this modified potential, (Ref. 4, p. 102),

$$\boldsymbol{\Omega} = \eta \hat{\nabla} \xi \tag{11}$$

The gradient $\hat{\nabla}\xi$ is still in the $\boldsymbol{\Omega}$ direction, so we may conclude as before that ξ = constant is a wall streamline. However, now the magnitude of $\hat{\nabla}\xi$ is modified by η to give the proper vorticity.

Corresponding to Eq. (9), we obtain a slightly more complicated final equation

$$\hat{\nabla} \cdot (\eta \hat{\nabla} \xi) = -\boldsymbol{n} \cdot (\mathrm{d}\omega/\mathrm{d}n)_{x_2=0} \tag{12}$$

A second equation relating η and ξ is found from the momentum Eq. (5),

$$\mu \boldsymbol{n} \cdot \hat{\nabla} \times (\eta \hat{\nabla} \xi) = -\boldsymbol{n} \cdot \nabla p|_{x_2=0} \tag{13}$$

We regard these equations as determining η and ξ for given distributions of normal vorticity flux and normal pressure gradient. In a full three-dimensional flow, both factors are required to fix the wall streamlines. In a boundary layer, the pressure gradient vanishes and $\eta = 1$ reduces Eq. (12) to the proper form. The elliptic natture of the ξ equation is still retained.

Summary

In summary, we have shown that it is possible to define streamfunction on the surface of solid bodies. Numerical values of the streamfunction are related to the vorticity and the shear stress on the surface. The differential equation which governs the streamfunction is elliptic and contains source terms which depend upon the adjacent flowfield. One of these terms is the normal vorticity flux at the wall ($\partial\omega_2/\partial x_2$) and the other is the pressure gradient normal to the wall ($\partial p/\partial x_2$). From another viewpoint the equation may be regarded as a compatability condition between the wall streamlines and the vorticity and pressure terms. It is not known whether this condition is actually important in the solution of three-dimensional boundary layers. It is noteworthy that the compatibility condition is elliptic, while the boundary-layer equations themselves are parabolic.

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